J. Fluid Mech. (1968), *vol.* 31, *part* 4, *pp.* 723–736
Printed in Great Britain

On the solution of the Boltzmann equation for an unsteady cylindrically symmetric expansion of a monatomic gas into a vacuum

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(Received 14 July 1967)

The problem of an unsteady axisymmetric expansion of a monatomic gas into a vacuum is considered in the limit of small source Knudsen number. It is shown that a solution of the Boltzmann equation for Maxwell molecules valid for large time can be constructed, which matches with the known equilibrium solution for an inviscid expansion of a fixed mass of gas into a vacuum provided that the region near the zero density front is excluded. This solution is formally the same as that obtained for the similar problem of steady spherical expansion into a vacuum—the variations along each particle path of the unsteady flow being the same as that in the steady flow.

Near the front, the expansion procedure breaks down and the equations require a different scaling. A modified form of the Boltzmann equation is obtained which leads to a corresponding set of moment equations. Unfortunately, the set of moment equations is no longer closed and no essential simplification has been made.

1. Introduction

Recent solutions of the Boltzmann equation for steady spherical expansion into a vacuum have shown that, by means of a suitable scaling, a closed set of moment equations can be derived for Maxwell molecules in the limit of large collision cross-section (Freeman 1967). This has enabled distributions of temperature to be obtained (Edwards & Cheng 1966; Hamel & Willis 1966) and, in principle, higher order moments of the distribution function to be calculated (Freeman & Thomas 1967). A similar set of moment equations can be computed for the unsteady, axisymmetric expansion into a vacuum and the two-dimensional steady expansion into a vacuum (Grundy 1967). It can be shown that these equations are again derivable by a suitable scaling of the Boltzmann equation in the case of Maxwell molecules.

In this paper, the flow of an unsteady, axisymmetric monatomic gas into a vacuum will be considered. It will be shown that, away from the vacuum-gas boundary (see figure 1), the equations of motion may be formally reduced to those for the steady spherical expansion into a vacuum along each particle path. This enables the complete solution for the spherical expansion to be used to construct this 'core' flow. Thus not only the temperature distribution but also higher order moments of the distribution function may be computed. Near the leading edge of the disturbance, however, the scaling breaks down and the solutions are not uniformly valid. A further scaling in this region shows that a closed set of moment equations cannot be obtained without some further simplifying assumptions.

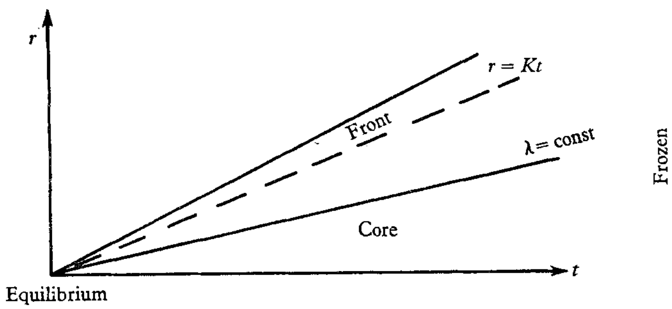

FIGURE 1. Development of the flow in the (r, t)-plane.

Mathematically, the procedure adopted in solving the problem is to consider a perturbation solution of the problem based on the already known equilibrium (i.e. inviscid) solution due to Sedov, Keller and Thornhill (see Mirels & Mullen 1963). Such a solution will break down for large times in the core region and a uniformly valid expansion must be sought. This leads to the formal identification of the behaviour along particle paths as being the same as that of a spherical steady problem—a result which had already been conjectured by Hamel & Willis (1966). The solution so obtained can then be matched with the equilibrium solution. In principle, further terms of this asymptotic expansion and the corresponding matching could be obtained, but this will not be attempted here.

The breakdown of the core solution as the front is approached is more complex, but a suitable scaling can be introduced and a further asymptotic expansion procedure evolved together with the appropriate matching scheme. The set of equations obtained for the second-order moments now contains third-order moments and the set of moment equations is no longer closed. Subsequent moments of the equation indicate that higher order moments than the one being sought will always intrude and consequently no essential simplification of the Boltzmann equation has been made. No further progress can be made in this region without some further assumption such as the adoption of a truncation procedure similar to that of Grad (1949).

2. Solution of the Boltzmann equation

The Boltzmann equation for unsteady, cylindrically symmetric flow is

$$\left[\frac{\partial}{\partial t}+\xi\frac{\partial}{\partial r}+\frac{\eta^2}{r}\frac{\partial}{\partial \xi}-\frac{\xi\eta}{r}\frac{\partial}{\partial \eta}\right]f=\left(\frac{\partial f}{\partial t}\right)_{\text{coll}}, \tag{2.1}$$

where polar co-ordinates r, θ and z are chosen with molecular velocities (ξ, η, ζ).

The right-hand side expresses the change in f due to collisions. This may be written (Chapman & Cowling 1960)

$$\left(\frac{\partial f}{\partial t}\right)_{\text{coll}} = \Lambda\int\ldots\int(f'f'_1-ff_1)\,d\xi_1 d\eta_1 d\zeta_1 v_0 dv_0 d\epsilon, \tag{2.2}$$

where $f_1 = f(\xi_1,\eta_1,\zeta_1)$ and $f' = f(\xi',\eta',\zeta')$ with $'$ denoting the velocities after collision, $\Lambda = (\kappa_{12}/2m)^{\frac{1}{2}}$ for a force law $P = \kappa_{12}/r^5$ between molecules of mass m. The molecules are thus assumed to be Maxwellian which obey the force law given above. For such molecules, the quantity v_0 is related to the velocities before and after collision by the collision mechanics (Chapman & Cowling 1960), but the integration over v_0 and ϵ is independent of that over the velocity field.

The equation may be re-written in a more convenient form if new variables $\rho^2 = \xi^2+\eta^2$ and $\alpha = r\eta$ are introduced whence

$$\frac{\partial f}{\partial t}+\left(\rho^2-\frac{\alpha^2}{r^2}\right)^{\frac{1}{2}}\frac{\partial f}{\partial r} = \left(\frac{\partial f}{\partial t}\right)_{\text{coll}}. \tag{2.3}$$

The macroscopic quantities number density and temperature may then be written as

$$N = \iiint f d\xi d\eta d\zeta$$

$$= \frac{1}{r}\iiint f\frac{\rho d\rho d\alpha d\zeta}{(\rho^2-\alpha^2/r^2)^{\frac{1}{2}}} \tag{2.4}$$

and

$$3NRT = \frac{1}{r}\iiint f(\rho^2+\zeta^2)\frac{\rho d\rho d\alpha d\zeta}{(\rho^2-\alpha^2/r^2)^{\frac{1}{2}}}. \tag{2.5}$$

The equilibrium flow associated with this equation is obtained by putting $(\partial f/\partial t)_{\text{coll}} = 0$, or, formally, as the limit $\Lambda\to\infty$. The solution is the Maxwell distribution function

$$f = \frac{N}{(2\pi RT)^{\frac{3}{2}}}\exp-\frac{(\xi-\bar{\xi})^2+\eta^2+\zeta^2}{2RT}, \tag{2.6}$$

where N, T and $\bar{\xi}$ are the local number density, temperature and gas speed respectively. In terms of the moment equations, this is the inviscid flow and the corresponding axisymmetric equations are well known. For the case of a fixed mass of gas expanding into a vacuum, a solution of these equations has been given by Sedov, Keller and Thornhill (see Mirels & Mullen 1963). The solution may be written in the form

$$N = (1-\lambda_1^2)^{\frac{3}{2}}/R^2, \quad \bar{\xi} = dR/dt, \tag{2.7}$$

where

$$\frac{dR}{dt} = \frac{3}{\sqrt{2}}(1-R^{-\frac{4}{3}})^{\frac{1}{2}} \tag{2.8}$$

and

$$\lambda_1 = r/R.$$

The number density is zero on the front $r = R$ or $\lambda_1 = 1$. It will be observed that the variables introduced here are non-dimensional and have been scaled in terms of source conditions. To obtain the dimensional form of these variables, they must be multiplied by the appropriate dimensional source variable. Thus, if [ ] denotes the dimensional variable we have

$$\left.\begin{aligned}[p] = pp_0, \quad [\rho] = \rho\rho_0, \quad [u] = u(p_0/\rho_0)^{\frac{1}{2}},\\ [r] = rL \quad \text{and} \quad [t] = tL/(p_0/\rho_0)^{\frac{1}{2}},\end{aligned}\right\} \tag{2.9}$$

where $L = (5M/2\pi\rho_0)$ and p_0, ρ_0 and M are the source pressure, density and mass respectively. Henceforth, the equations will be described solely in terms of these non-dimensional variables.

For large r and t, but such that r/t remains finite, the solution becomes

$$N = \frac{(1-\lambda_2^2)^{\frac{3}{2}}}{K^2t^2}, \quad \bar{\xi} = K\lambda_2 = \frac{r}{t} \tag{2.10}$$

and $R = Kt$, where $K = 3/\sqrt{2}$ and $\lambda_2 = r/Kt$.

If this behaviour is used as a first approximation for large Λ and an asymptotic expansion sought in inverse powers of Λ, as $t \to \infty$ subsequent terms break down like t/Λ, r/Λ as in the case of the spherical steady expansion. This non-uniformity thus causes us to seek a uniformly valid expansion for Λ and t large.

We now write the Boltzmann equation in the form

$$\frac{\partial f}{\partial t} + \left(\rho^2 - \frac{\alpha^2}{r^2}\right)^{\frac{1}{2}} \frac{\partial f}{\partial r} = ANI, \tag{2.11}$$

where A is the non-dimensional form of Λ and $(\partial f/\partial t)_{\text{coll}} = ANI$.

The quantity NI denotes the collision integral and is, in general, a function of ρ, α, ζ, r and t. Again, this equation is written in non-dimensional form and hence the quantity A is non-dimensional. It may be regarded as an inverse source Knudsen number. For molecules distributed according to the Maxwell distribution, we may write $A = (2/\sqrt{\pi})(L/l)$ where l is the molecular mean free path. The limiting process of interest here is the limit $A \to \infty$. In order that r/A, t/A remain of order unity in the limit we must introduce new variables

$$s = r/\text{A} \quad \text{and} \quad t_1 = t/A. \tag{2.12}$$

A corresponding scaling in the other variables can easily be obtained as

$$\left.\begin{aligned} n &= NA^2, & \psi &= \alpha A^{-\frac{1}{3}},\\ \tau &= TA^{\frac{4}{3}}, & \chi &= \zeta A^{\frac{2}{3}},\\ \phi &= (\rho - \bar{\xi})\, A^{\frac{2}{3}}. \end{aligned}\right\} \tag{2.13}$$

Introducing these into (2.11), we obtain

$$\frac{\partial f}{\partial t_1} + \bar{\xi}\frac{\partial f}{\partial s} - \phi\frac{\partial f}{\partial \phi}\frac{\partial \bar{\xi}}{\partial s} - A^{\frac{2}{3}}\frac{\partial f}{\partial \phi}\left[\frac{\partial \bar{\xi}}{\partial t_1} + \bar{\xi}\frac{\partial \bar{\xi}}{\partial s}\right] + A^{-\frac{2}{3}}\left(\phi\frac{\partial f}{\partial s} - \frac{\psi^2}{2\bar{\xi}s^2}\frac{\partial \bar{\xi}}{\partial s}\frac{\partial f}{\partial \phi}\right)$$
$$= n(I(\phi, \psi, \chi, s, t_1) + A^{-\frac{2}{3}}I_1) + O(A^{-\frac{4}{3}}), \tag{2.14}$$

where I_1 denotes the first term in the expansion of I. It is not necessary at this stage to give all the detail of equation (2.14) but this will be found necessary later (§3).

Multiplication of (2.14) by ϕ and integration over the velocity space gives the momentum equation in the r-direction as

$$\frac{\partial\bar{\xi}}{\partial t_1}+\bar{\xi}\frac{\partial\bar{\xi}}{\partial s} = O(A^{-\frac{4}{3}}). \tag{2.15}$$

The order of the error here should be noted. It is smaller than might be expected, since all the order one terms in (2.14) vanish when this moment is taken. This equation shows that $\bar{\xi}$ is given by the hypersonic approximation.

Inserting this back in (2.14) then reduces the equation to

$$\frac{\partial f}{\partial t_1}+\bar{\xi}\frac{\partial f}{\partial s}-\phi\frac{\partial f}{\partial \phi}\frac{\partial\bar{\xi}}{\partial s} = nI \tag{2.16}$$

to first order. The terms neglected are of order $A^{-\frac{2}{3}}$. The number density and temperature become

$$\left.\begin{aligned} n &= \frac{1}{s}\iiint f d\phi d\psi d\chi, \\ \text{and}\qquad 3n\tau &= \frac{1}{s}\iiint\left(\phi^2+\frac{\psi^2}{s^2}+\chi^2\right) f d\phi d\psi d\chi. \end{aligned}\right\} \tag{2.17}$$

Comparison of the solution of (2.15) with the equilibrium solution shows that

$$\bar{\xi} = s/t_1. \tag{2.18}$$

Equation (2.16) then becomes

$$\frac{\partial f}{\partial t_1}+\frac{s}{t_1}\frac{\partial f}{\partial s}-\frac{\phi}{t_1}\frac{\partial f}{\partial \phi} = nI. \tag{2.19}$$

It is now more convenient to work in variables

$$\Phi = \phi t_1, \quad \Psi = \psi/\lambda, \quad \lambda = s/t_1 \quad \text{and} \quad t_1 \text{ itself.} \tag{2.20}$$

The equation (2.19) reduces to

$$\left(\frac{\partial f}{\partial t_1}\right) = nI(\Phi, \Psi, \chi, \lambda, t_1) \tag{2.21}$$

in these variables, with

$$n = \frac{1}{t_1^2}\iiint f d\Phi d\Psi d\chi$$

and

$$3n\tau = \frac{1}{t_1^2}\iiint\left(\frac{\Phi^2}{t_1^2}+\frac{\Psi^2}{t_1^2}+\chi^2\right) d\Phi d\Psi d\chi.$$

The derivative is taken at constant Φ, Ψ, χ and λ. It should be noted that

$$\Psi = \psi/\lambda = A^{-\frac{1}{3}}\eta t = \eta t_1 A^{\frac{2}{3}}, \tag{2.22}$$

and hence η and $\rho-\bar{\xi}$ are now scaled in exactly the same way. This result could have been obtained by choosing Ψ as a variable initially. It would, however, have complicated the initial development of the equations.

The similarity of the above set of equations to the spherically symmetric equations of steady flow into a vacuum (Freeman 1967) should be noted. If the time t, is replaced by $r/\bar{\xi}$ and a suitable cyclic permutation of the velocity coordinates introduced, then these equations are identical with the equations for spherically symmetric flow. Several important conclusions may be drawn from this similarity. First, it will be observed that Φ and Ψ appear in a similar way in the problem, since the collision integral cannot depend on the choice of velocity co-ordinates. This immediately leads to the conclusion that moments in the r and θ directions will be identical. Hence the r component of stress P_{rr} will equal the θ component $P_{\theta\theta}$. This corresponds in the spherically symmetric steady problem to the more obvious identity of $P_{\theta\theta}$ and $\bar{P}_{\phi\phi}$. Secondly, in the cylindrical problem χ plays the same role as the r-component of velocity in the spherical case. We would thus expect that the z-component of the temperature will freeze in this case.

It is interesting to note that the variable λ does not feature either in the differentiation or integration of equation (2.21). Hence, it will behave simply as a parameter in the problem. Its inclusion in the non-equilibrium solution will occur in the form suggested by matching the solutions of equation (2.21) with the equilibrium solution.

It is necessary, in order to obtain a solution of equation (2.21), to deduce the moment equations. The first moment equation obtained by integrating (2.21) over the velocity space is

$$\frac{\partial}{\partial t_1}(nt_1^2) = 0, \tag{2.23}$$

which may be integrated directly to give

$$n = (1-\lambda^2/K^2)^{\frac{3}{2}}/K^2t_1^2, \tag{2.24}$$

where the arbitrary function of λ has been evaluated by comparison with the equilibrium solution.

The equations of second-order moments may be obtained from (2.21) as the closed set (Freeman 1967)

$$\frac{\partial}{\partial t_1}\overline{\chi}^2 = \frac{Bnt_1^2}{t_1^2}\left[\frac{\overline{\Omega}^2}{2t_1^2} - \overline{\chi}^2\right] \tag{2.25}$$

and

$$\frac{\partial}{\partial t}\overline{\Omega}^2 = Bnt_1^2\left[\overline{\chi}^2 - \frac{\overline{\Omega}^2}{2t_1^2}\right],$$

where

$$\Omega^2 = \Phi^2+\Psi^2, \quad \overline{\Omega}^2 = \frac{\iint \Omega^2 f\Omega\, d\Omega\, d\chi}{\iint f\Omega\, d\Omega\, d\chi} \quad \text{and} \quad B = \pi\int \sin^2\chi v_0\, dv_0 = 1{\cdot}371,$$

a constant evaluated from the collision mechanics. If initially we had written A_1 for the quantity $A(Bnt_1^2)$, then the scaling would have proceeded in a similar way with A_1 replacing A and the constant in (2.25) would have been absent.

We see that

$$\begin{aligned} 3\tau &= \frac{2\pi}{nt_1^2}\iint\left(\frac{\Omega^2}{t_1^2}+\chi^2\right)f\Omega\, d\Omega\, d\chi \\ &= 2\pi\frac{\iint\left(\frac{\Omega^2}{t_1^2}+\chi^2\right)f\Omega\, d\Omega\, d\chi}{2\pi\iint f\Omega\, d\Omega\, d\chi} \\ &= \frac{\overline{\Omega}^2}{t_1^2}+\overline{\chi}^2. \end{aligned} \tag{2.26}$$

The equation for τ from (2.25) is

$$t_2^2 \frac{\partial \tau}{\partial t_2^2} + (3t_2 + 1)\frac{\partial \tau}{\partial t_2} + \frac{4}{3}\frac{\tau}{t_2} = 0, \tag{2.27}$$

where $t_2 = 2t_1/3Bnt_1^2$.

This is the equation for the temperature obtained in the spherically symmetric case. The solution of (2.27) appropriate to this problem is

$$\tau = D\psi(-\tfrac{4}{3}, -1; 1/t_2), \tag{2.28}$$

where ψ is the confluent hypergeometric function (Erdélyi 1953). D is a constant of the differentiation and hence depends only on λ. Now, in equilibrium, we have

$$T = (N/N_0)^{\frac{2}{3}}$$

$$= (1-\lambda_2^2)/(Kt)^{\frac{4}{3}},$$

or

$$\tau = (1-\lambda^2/K^2)/(Kt_1)^{\frac{4}{3}}. \tag{2.29}$$

Comparing this with the behaviour of (2.28) for t_2 small which is $Dt_2^{\frac{4}{3}}$ we obtain

$$D = 2^{\frac{2}{3}}/B^{\frac{4}{3}}(1-\lambda^2/K^2), \tag{2.30}$$

whence

$$\tau = \left[2^{\frac{2}{3}}/B^{\frac{4}{3}}\left(1-\frac{\lambda^2}{K^2}\right)\right]\psi\left(-\frac{4}{3}, -1; \frac{3B(1-\lambda^2/K^2)^{\frac{3}{2}}}{2K^2 t_1}\right). \tag{2.31}$$

In a similar way the various stress components may be obtained by integrating (2.25). Further, higher order moments may be computed as in the case of the steady spherically symmetric solution (Freeman & Thomas 1967) and matching with the equilibrium solutions allows a complete solution to be determined.

In general, if the solution to the spherically symmetric steady problem for some moment is

$$\overline{\Gamma} = G(s),$$

which has an equilibrium variation $G(s) \sim s^\alpha$, then the corresponding core solution for the unsteady cylindrical flow is given by

$$\overline{\Gamma} = H(\lambda)\left(\frac{3B(1-\lambda^2/K^2)^{\frac{3}{2}}}{2K^2}\right)^\alpha G\left(\frac{2K^2 t_1}{3B(1-\lambda^2/K^2)^{\frac{3}{2}}}\right),$$

where the equilibrium value of $\overline{\Gamma}$ for cylindrical flow is $H(\lambda)t_1^\alpha$.

It is apparent, however, that difficulties arise with the temperature when $\lambda = K$, i.e. at the front. We see that for fixed λ, the temperature becomes constant as $t_1 \to \infty$ and has the 'frozen' value

$$\tau_F = \frac{2^{\frac{2}{3}}}{B^{\frac{4}{3}}(1-\lambda^2/K^2)}\frac{\Gamma(2)}{\Gamma(\frac{2}{3})}.$$

As the front is approached, however, this becomes infinite. At fixed t_1, the solution is infinite at the front, although the singular nature of the expansion allows the correct zero value to be achieved as equilibrium is approached. This behaviour is, of course, only another indication of the singular nature of all the solutions in the front region. It is obvious that in this region, the scaling introduced earlier is no longer capable of describing the behaviour of the equations. The nature of the difficulty is, however, apparent from the manner of the breakdown. It will be necessary to introduce a suitable scaling for $1-\lambda^2/K^2$ in this region to overcome this difficulty.

The variation of τ along lines of constant λ is shown in figure 2. The variation of τ across the flow is indicated in figure 3. The breakdown of the theory as $\lambda/K \to 1$ results in the rapid variation of τ at the front.

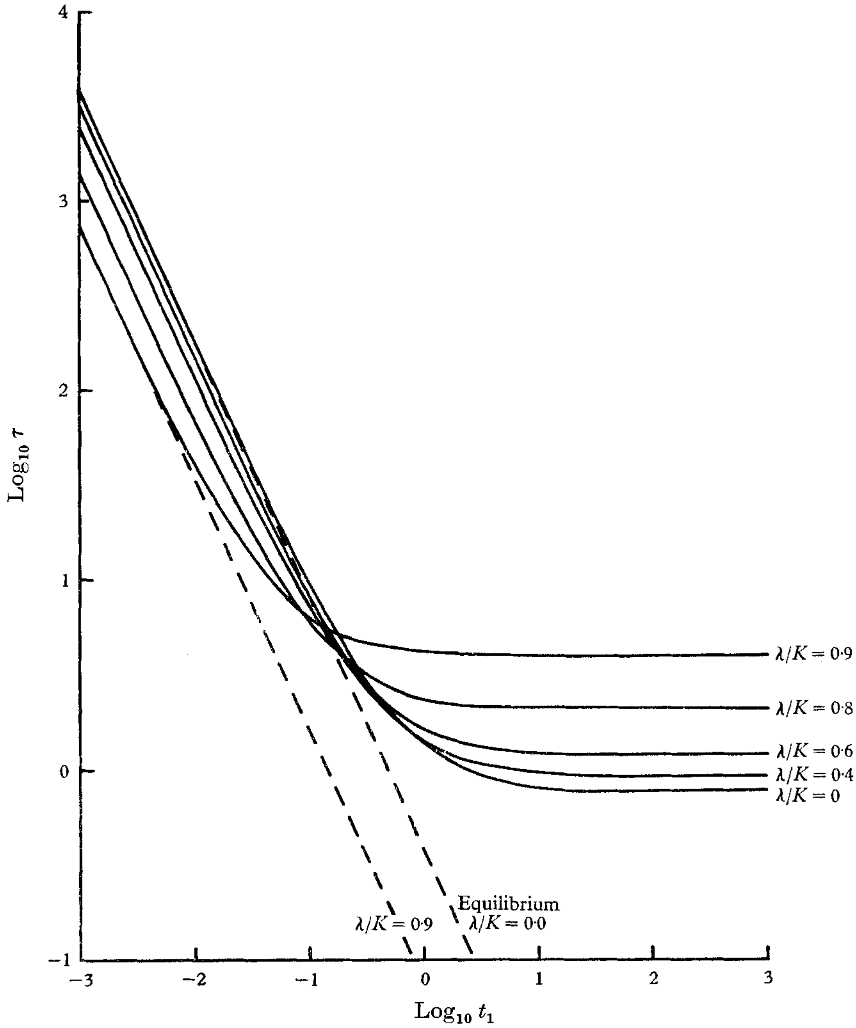

FIGURE 2. Variation of temperature along particle paths: – – –, equilibrium; ——, non-equilibrium.

3. Solution near the front

Near the front $\lambda = K$, so that $(1-\lambda^2/K^2)$ tends to zero. The manner of the breakdown indicates that this quantity is, in fact, of order $A^{-\frac{4}{9}}$ near the front. Consequently, a new set of variables must be introduced as follows:

$$\left.\begin{aligned} \lambda' &= A^{\frac{4}{9}}(\lambda - K), & \chi' &= A^{-\frac{2}{9}}\chi, \\ v' &= A^{\frac{4}{9}}(\bar{\xi} - K), & \tau' &= A^{\frac{4}{9}}\tau, \\ \phi' &= A^{-\frac{2}{9}}\phi, & n' &= A^{-\frac{2}{3}}n, \\ \psi' &= A^{\frac{4}{9}}\psi, & t' &= A^{\frac{2}{3}}t_1. \end{aligned}\right\} \qquad (3.1)$$

Re-writing the Boltzmann equation (2.14) in terms of t_1 and λ and then substituting the variables, (3.1) gives as a first approximation

$$\frac{\partial f}{\partial t'}+\frac{V-\lambda'}{t'}\frac{\partial f}{\partial \lambda'}-\frac{\phi'}{t'}\frac{\partial f}{\partial \phi'}\frac{\partial V}{\partial \lambda'}-\frac{\partial f}{\partial \phi'}\left[\frac{\partial V}{\partial t'}+\frac{V-\lambda'}{t'}\frac{\partial V}{\partial \lambda'}\right]+\frac{\phi'}{t'}\frac{\partial f}{\partial \lambda'} = n'I(\phi',\psi',\chi',\lambda',t'), \tag{3.2}$$

with
$$n' = \frac{1}{Kt'}\iiint f d\phi' d\psi' d\chi'$$

and
$$3n'\tau' = \frac{1}{Kt'}\iiint\left(\phi'^2+\frac{\psi'^2}{K^2t'^2}+\chi'^2\right) f d\phi' d\psi' d\chi'.$$

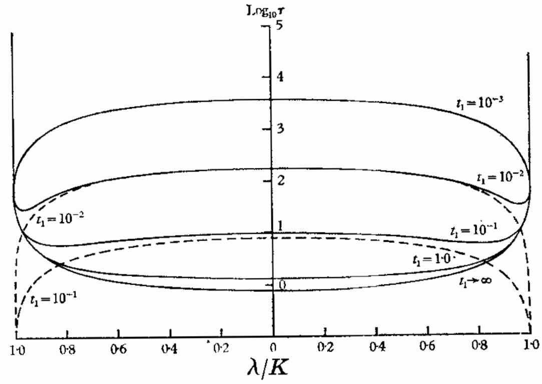

FIGURE 3. Variation of temperature at fixed time: ---, equilibrium; ——, non-equilibrium.

In order to make any progress with the solution of this equation, it is necessary to consider the moment equations. Up to second order these are

$$\left.\begin{aligned}
&\frac{\partial}{\partial t'}(n't')+\frac{V-\lambda'}{t'}\frac{\partial}{\partial \lambda'}(n't')+n'\frac{\partial V}{\partial \lambda'} = 0,\\
&n't'\left[\frac{\partial V}{\partial t}+\frac{V-\lambda'}{t'}\frac{\partial V}{\partial \lambda'}\right]+\frac{\partial P_{rr}}{\partial \lambda} = 0,\\
&\frac{\partial}{\partial t'}(t'P_{rr})+(V-\lambda')\frac{\partial}{\partial \lambda'}P_{rr}+3P_{rr}\frac{\partial V}{\partial \lambda'}+\frac{\partial}{\partial \lambda'}Q_{rrr} = B_1 n't'(n'\tau'-P_{rr}),\\
&\frac{\partial}{\partial t'}(t'^3P_{\theta\theta})+(V-\lambda')\frac{\partial}{\partial \lambda'}(t'^2P_{\theta\theta})+t'^2P_{\theta\theta}\frac{\partial V}{\partial \lambda'}+\frac{\partial}{\partial \lambda'}Q_{r\theta\theta}\\
&\qquad = B_1 n't'^3(n'\tau'-P_{\theta\theta}),\\
&\frac{\partial}{\partial t'}(t'P_{zz})+(V-\lambda')\frac{\partial}{\partial \lambda'}P_{zz}+P_{zz}\frac{\partial V}{\partial \lambda'}+\frac{\partial}{\partial \lambda'}Q_{rzz} = B_1 n't'(n'\tau'-P_{zz}),
\end{aligned}\right\} \tag{3.3}$$

where
$$P_{rr} = \frac{1}{Kt'}\iiint \phi'^2 f d\phi' d\psi' d\chi',$$
$$P_{\theta\theta} = \frac{1}{(Kt')^3}\iiint \psi'^2 f d\phi' d\psi' d\chi',$$
$$P_{zz} = \frac{1}{Kt'}\iiint \chi'^2 f d\phi' d\psi' d\chi',$$
$$n'\tau' = \tfrac{1}{3}(P_{rr}+P_{\theta\theta}+P_{zz}),$$

$$Q_{rrr} = \frac{1}{Kt'}\iiint \phi'^3 f d\phi' d\psi' d\chi',$$

$$Q_{r\theta\theta} = \frac{1}{(Kt')^3}\iiint \phi'\psi'^2 f d\phi' d\psi' d\chi',$$

$$Q_{rzz} = \frac{1}{Kt'}\iiint \phi'\chi'^2 f d\phi' d\psi' d\chi',$$

and $$B_1 = \tfrac{3}{2}B.$$

It is apparent that the simplification of the equations which occurred in the core region is not possible here. In particular, it will be obvious that the moment equations will not form a closed set of equations due to the presence of the final term on the right-hand side of (3.2). The second-order moment equations in (3.3) now contain third-order moments. Further moment equations to deduce the third-order moments will necessarily include fourth-order moments and so on. It is therefore impossible to obtain a solution of the system so derived without using some further procedure of approximation such as the truncation after the third-order moments suggested by Grad (1949). We will, however, consider the relationship of equations (3.3) with the core solution further in §§4 and 5.

4. Solution for small time

A solution of the equations at the front may be derived by introducing an asymptotic expansion for small time of the form:

$$\left.\begin{aligned} n' &= \frac{1}{t'^4}[F_0(\Lambda)+t'^{\frac{5}{3}}F_1(\Lambda)+\ldots],\\ V &= \frac{1}{t'^{\frac{4}{3}}}[U_0(\Lambda)+t'^{\frac{5}{3}}U_1(\Lambda)+\ldots],\\ P'_{rr} &= \frac{1}{t'^{\frac{20}{3}}}[P_{rr0}(\Lambda)+t'^{\frac{5}{3}}P_{rr1}(\Lambda)+\ldots],\end{aligned}\right\} \tag{4.1}$$

and similarly for $P_{\theta\theta}$ and P_{zz}

$$Q'_{rrr} = \frac{1}{t'^{\frac{28}{3}}}[Q_{rrr1}(\Lambda)+t'^{\frac{4}{3}}Q_{rrr2}(\Lambda)+\ldots]$$

and similarly for $Q_{r\theta\theta}$ and Q_{rzz}

$$\tau' = \frac{1}{t'^{\frac{8}{3}}}[S_0(\Lambda)+t'^{\frac{5}{3}}S_1(\Lambda)+\ldots],$$

where $$\Lambda = t'^{\frac{4}{3}}\lambda'.$$

Substituting in the differential equations gives, to first order,

$$(U_0+\tfrac{1}{3}\Lambda)\frac{dF_0}{d\Lambda}+\left[\frac{dU_0}{d\Lambda}-3\right]F_0 = 0,$$

$$(U_0+\tfrac{1}{3}\Lambda)\frac{dU_0}{d\Lambda}-\tfrac{4}{3}U_0+\frac{1}{F_0}\frac{dP_{rr0}}{d\Lambda} = 0 \tag{4.2}$$

and $$P_{rr0} = P_{\theta\theta 0} = P_{zz0} = F_0 S_0 \quad (=P_0\text{, say}).$$

A further equation may be obtained by adding the three rate equations to obtain an energy equation. Taking account only of first-order terms in this equation gives

$$(U_0+\tfrac{1}{3}\Lambda)\frac{d}{d\Lambda}(F_0S_0)+\tfrac{5}{3}F_0S_0\frac{dU_0}{d\Lambda}-5F_0S_0 = 0. \tag{4.3}$$

Re-writing this equation gives

$$\frac{1}{P_0}\frac{dP_0}{d\Lambda}+\frac{5}{3U_0+\Lambda}\left(\frac{dU_0}{d\Lambda}-3\right) = 0. \tag{4.4}$$

A combination of (4.2) and (4.4) then gives

$$\frac{1}{P_0}\frac{dP_0}{d\Lambda} = \frac{5}{3}\frac{1}{F_0}\frac{dF_0}{d\Lambda}, \tag{4.5}$$

or $Q_0 = CF^{\frac{5}{3}}$, where C is an arbitrary constant.

The equations (4.2) then become

$$\left.\begin{aligned}\frac{1}{F_0}\frac{dF_0}{d\Lambda} &= -\frac{3}{3U_0+\Lambda}\left(\frac{dU_0}{d\Lambda}-3\right),\\ (U_0+\tfrac{1}{3}\Lambda)\frac{dU_0}{d\Lambda}-\tfrac{4}{3}U_0+\frac{5}{2}\frac{dF_0^{\frac{2}{3}}}{d\Lambda} &= 0.\end{aligned}\right\} \tag{4.6}$$

A solution of these equations is

$$\left.\begin{aligned}F_0 = (3-2K^{\frac{1}{3}}\Lambda)^{\frac{3}{2}}/K^4, \quad U_0 &= \Lambda-2/K^{\frac{1}{3}},\\ P_0 = (3-2K^{\frac{1}{3}}\Lambda)^{\frac{5}{2}}/K^{\frac{20}{3}} \quad \text{and} \quad S_0 &= (3-2K^{\frac{1}{3}}\Lambda)/K^{\frac{8}{3}}.\end{aligned}\right\} \tag{4.7}$$

These are in fact the first-order equilibrium solutions when scaled in the appropriate variables at the front. This form of expansion thus describes the near equilibrium behaviour of the front equations.

It should be noted that the error terms arise solely from the heat transfer terms Q_{rrr}, etc., and consequently these terms will appear in the calculations of further terms in the expansion. It is difficult therefore to see how these higher order terms can be computed without resorting to an infinite set of higher order moment equations.

5. Solution for large time

For large time, an asymptotic expansion of the following form may be sought:

$$\left.\begin{aligned}n' &= \frac{1}{t'}(\mathscr{F}_0(\Lambda^*)+t'^{-\frac{4}{3}}\mathscr{F}_1(\Lambda^*)+\ldots),\\ V &= t'^{-\frac{2}{3}}(\mathscr{U}_0(\Lambda^*)+t'^{-\frac{4}{3}}\mathscr{U}_1(\Lambda^*)+\ldots),\\ P'_{rr} &= t'^{-\frac{5}{3}}(\mathscr{P}_{rr0}(\Lambda^*)+t'^{-\frac{4}{3}}\mathscr{P}_{rr1}(\Lambda^*)+\ldots),\end{aligned}\right\} \tag{5.1}$$

and similarly for $P_{\theta\theta}$ and P_{zz},

$$Q'_{rrr} = t'^{-\frac{7}{3}}[\mathscr{Q}_{rrr1}(\Lambda^*)+t'^{-\frac{4}{3}}\mathscr{Q}_{rrr2}+\ldots],$$

and similarly for $Q_{r\theta\theta}$ and Q_{rzz},

$$\tau' = t'^{-\frac{2}{3}}[\mathscr{S}_0(\Lambda^*)+t'^{-\frac{4}{3}}\mathscr{S}_1(\Lambda^*)+\ldots],$$

where
$$\Lambda^* = \lambda' t'^{-\frac{2}{3}}.$$

Substituting in (3.3) we obtain

$$\left.\begin{aligned}
&(\mathcal{U}_0-\tfrac{5}{3}\Lambda^*)\frac{d\mathcal{F}_0}{d\Lambda^*}+\mathcal{F}_0\frac{d\mathcal{U}_0}{d\Lambda^*}=0,\\
&(\mathcal{U}_0-\tfrac{5}{3}\Lambda^*)\frac{d\mathcal{U}_0}{d\Lambda^*}+\tfrac{2}{3}\mathcal{U}_0=0,\\
&(\mathcal{U}_0-\Lambda^*)\frac{d}{d\Lambda^*}\mathcal{P}_{rr0}+3\mathcal{P}_{rr0}\frac{d\mathcal{U}_0}{d\Lambda^*}-\frac{2}{3}\frac{d}{d\Lambda^*}(\mathcal{P}_{rr0}\Lambda^*)=\mathcal{F}_0(\mathcal{F}_0\mathcal{S}_0-\mathcal{P}_{rr0}),\\
&(\mathcal{U}_0-\Lambda^*)\frac{d}{d\Lambda^*}\mathcal{P}_{\theta\theta0}+\mathcal{P}_{\theta\theta0}\frac{d\mathcal{U}_0}{d\Lambda^*}+\tfrac{4}{3}\mathcal{P}_{\theta\theta0}-\tfrac{2}{3}\Lambda^*\frac{d}{d\Lambda^*}\mathcal{P}_{\theta\theta0}\\
&\qquad=\mathcal{F}_0(\mathcal{F}_0\mathcal{S}_0-\mathcal{P}_{\theta\theta0})\\
\text{and}\quad&(\mathcal{U}_0-\Lambda^*)\frac{d}{d\Lambda^*}\mathcal{P}_{zz0}+\mathcal{P}_{zz0}\frac{d}{d\Lambda^*}\mathcal{U}_0-\frac{2}{3}\frac{d}{d\Lambda}(\mathcal{P}_{zz0}\Lambda^*)\\
&\qquad=\mathcal{F}_0(\mathcal{F}_0\mathcal{S}_0-\mathcal{P}_{zz0}).
\end{aligned}\right\}\tag{5.2}$$

The general solution of the second of equations (5.2) can be obtained parametrically as

$$\mathcal{U}_0=W^{\frac{2}{3}}(1+CW)^{-\frac{2}{3}},\quad \Lambda^*=W^{\frac{2}{3}}(1+CW)^{-\frac{5}{3}},\tag{5.3}$$

where C is an arbitrary constant.

The matching equations require that

$$C=0,$$

and the solution becomes

$$\mathcal{U}_0=\Lambda^*.\tag{5.4}$$

Substituting into the first equation gives

$$\frac{\mathcal{F}_0'}{\mathcal{F}_0}=\frac{3}{2}\frac{1}{\Lambda^*},\quad\text{or}\quad \mathcal{F}_0=\Lambda^{*\frac{3}{2}},$$

where the matching with the core solution has again been made. Using these results the final three equations become

$$\left.\begin{aligned}
&3\mathcal{P}_{rr0}-\frac{2}{3}\frac{d}{d\Lambda^*}(\mathcal{P}_{rr0}\Lambda^*)=\Lambda^{*\frac{3}{2}}(\Lambda^{*\frac{3}{2}}\mathcal{S}_0-\mathcal{P}_{rr0}),\\
&3\mathcal{P}_{\theta\theta0}-\frac{2}{3}\frac{d}{d\Lambda^*}(\mathcal{P}_{\theta\theta0}\Lambda^*)=\Lambda^{*\frac{3}{2}}(\Lambda^{*\frac{3}{2}}\mathcal{S}_0-\mathcal{P}_{\theta\theta0})
\end{aligned}\right\}\tag{5.5}$$

and

$$\mathcal{P}_{zz0}-\frac{2}{3}\frac{d}{d\Lambda^*}(\mathcal{P}_{zz0}\Lambda^*)=\Lambda^{*\frac{3}{2}}(\Lambda^{*\frac{3}{2}}\mathcal{S}_0-\mathcal{P}_{zz0}).\tag{5.6}$$

The equations (5.6) correspond to the limiting value of the core equations as the front is approached, though, written in terms of the similarity variable Λ^*, they are not easily recognized. The derivatives here correspond to a differentiation with respect to $t'^{-\frac{2}{3}}$ and not time itself and also there is a slight difference between the dependent variables introduced above and those in the core region. To see the correspondence, a new set of variables must be introduced. If we write

$$\Lambda^*=Y^{-\frac{2}{3}},\quad \mathcal{P}_{rr0}=\mathcal{R}Y^{-\frac{5}{3}},\ \text{etc.}\tag{5.7}$$

and

$$\mathcal{S}_0=\mathcal{T}Y^{\frac{2}{3}},$$

then these equations become

$$\left.\begin{aligned} Y\frac{d\mathscr{R}_{rr}}{dY}+4\mathscr{R}_{rr} &= \frac{1}{Y}\left(\frac{\mathscr{T}}{Y^2}-\mathscr{R}_{rr}\right),\\ Y\frac{d\mathscr{R}_{\theta\theta}}{dY}+4\mathscr{R}_{\theta\theta} &= \frac{1}{Y}\left(\frac{\mathscr{T}}{Y^2}-\mathscr{R}_{\theta\theta}\right),\\ Y\frac{d}{dY}\mathscr{R}_{zz}+2\mathscr{R}_{zz} &= \frac{1}{Y}\left(\frac{\mathscr{T}}{Y^2}-\mathscr{R}_{zz}\right). \end{aligned}\right\} \tag{5.8}$$

Using the relation

$$\mathscr{R}_{rr}+\mathscr{R}_{\theta\theta}+\mathscr{R}_{zz} = 3\mathscr{T}/Y^2,$$

we obtain

$$Y^2\frac{d^2\mathscr{T}}{dY^2}+\frac{d\mathscr{T}}{dY}(3Y+1)+\frac{4}{3}\frac{\mathscr{T}}{Y} = 0,$$

which is equation (2.27) for the temperature in the core region. This confirms that the zeroth order terms of the large time equation correspond to the limiting form of the core solutions at the front.

The error terms again arise from the heat transfer terms Q'_{rrr}, etc. and consequently these will occur in the next order solution making it difficult, if not impossible, to evaluate the higher order terms as in §4.

Neither of the similarity solutions described in §§4 and 5 brings us any nearer extending the solution to the edge of the core region $\lambda' = 0$ and beyond.

6. Conclusions

By use of a scaling within the core region, an analytic solution has been obtained from the closed set of moment equations up to second order. The similarity of this solution to the corresponding one for the steady spherical expansions indicates how higher order moments could be computed, and a direct transformation from the results obtained in one problem to that in the other can be stated. The dramatic breakdown of the solution in the front region, as is obvious in the computed results of figure 3, makes it imperative that new scaling be introduced. Although this is possible, the resulting equation for the distribution function has still the major difficulties of the Boltzmann equation contained within it. No closed system of moment equations is now possible, unless some other assumptions are employed to truncate the expansion scheme.

The authors would like to thank Dr P. A. Blythe for many useful discussions during the course of the preparation of this paper.

One of the authors (R. E. G.) was in receipt of an S.R.C. scholarship while engaged on this work.

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